

Flavor Hierarchy From F-theory

Jonathan J. Heckman* and Cumrun Vafa†

Jefferson Physical Laboratory, Harvard University, Cambridge, MA 02138, USA

It has recently been shown that F-theory based constructions provide a potentially promising avenue for engineering GUT models which descend to the MSSM. In this note we show that in the presence of background fluxes, these models automatically achieve hierarchical Yukawa matrices in the quark and lepton sectors. At leading order, the existence of a $U(1)$ symmetry which is related to phase rotations of the internal holomorphic coordinates at the brane intersection point leads to rank one Yukawa matrices. Subleading corrections to the internal wave functions from variations in the background fluxes generate small violations of this $U(1)$, leading to hierarchical Yukawa structures reminiscent of the Froggatt-Nielsen mechanism. The expansion parameter for this perturbation is in terms of $\sqrt{\alpha_{GUT}}$. Moreover, we naturally obtain a hierarchical CKM matrix with $V_{12} \sim V_{21} \sim \varepsilon$, $V_{23} \sim V_{32} \sim \varepsilon^2$, $V_{13} \sim V_{31} \sim \varepsilon^3$, where $\varepsilon \sim \sqrt{\alpha_{GUT}}$, in excellent agreement with observation.

I. INTRODUCTION

Flavor physics remains a poorly understood aspect of string based constructions as well as phenomenological models which aim to reproduce the Standard Model or MSSM at low energies. In this note we show that the minimal form of the F-theory GUT models recently developed in [1, 2, 3] (see also [4, 5, 6, 7, 8, 9, 10]) *automatically* generates viable Yukawa matrices in the quark and lepton sectors. In the present context, a minimal implementation of an F-theory GUT simply reflects the discrete choice in the geometry that we consider models with the fewest possible number of matter curves and intersection points necessary for compatibility with the MSSM.

The MSSM superpotential contains the terms:

$$W_{MSSM} = \lambda_u^{ij} \cdot Q^i U^j H_u + \lambda_d^{ij} \cdot Q^i D^j H_d + \lambda_l^{ij} \cdot L^i E^j H_d + \dots \quad (1)$$

where i and j are generation indices so that U^3 refers to the right-handed top quark, and the λ 's denote 3×3 matrices. Although genericity arguments might suggest all of the associated masses should be comparable, there is a well-known hierarchy [11]:

$$(m_u, m_c, m_t) \sim (0.003, 1.3, 170) \times \text{GeV} \quad (2)$$

$$(m_d, m_s, m_b) \sim (0.005, 0.1, 4) \times \text{GeV} \quad (3)$$

$$(m_e, m_\mu, m_\tau) \sim (0.0005, 0.1, 1.8) \times \text{GeV}. \quad (4)$$

To leading order, the inter-generational ratios of masses between particles with the same gauge quantum numbers are insensitive to running effects. On the other hand, running to the GUT scale reduces the heaviest generation quark masses by roughly a factor of three.

Letting $V_{u,d}^{L,R}$ denote unitary matrices such that $V_u^L \lambda_u V_u^{R\dagger}$ and $V_d^L \lambda_d V_d^{R\dagger}$ are diagonal, the norm of elements in the CKM matrix $V_{CKM} \equiv V_u^L V_d^{L\dagger}$ are also hierarchical [11]:

$$|V_{CKM}(M_{weak})| \sim \begin{pmatrix} 0.97 & 0.23 & 0.004 \\ 0.23 & 0.97 & 0.04 \\ 0.008 & 0.04 & 0.99 \end{pmatrix}. \quad (5)$$

This is encapsulated in the Wolfenstein parameterization of V_{CKM} [12]:

$$V_{CKM} = \begin{pmatrix} 1 - \lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \lambda^2/2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} + O(\lambda^4) \quad (6)$$

*Electronic address: jheckman@fas.harvard.edu

†Electronic address: vafa@physics.harvard.edu

where $\lambda \sim 0.23$, $A \sim 0.82$, $\rho \sim 0.22$, $\eta \sim 0.34$ at the weak scale. To one loop order, only A evolves with scale [13, 14]. In fact, even including two loop running effects, A only changes by an order one factor [15].

This type of hierarchy is potentially present in models with a global $U(1)$ flavor symmetry as in the Froggatt-Nielsen mechanism [16]. For example, assigning FN charges 4, 2, 0 to the three generations of Q , U superfields and introducing a chiral superfield X_{FN} with FN charge -1 , the superpotential term $(X_{FN}/M_{FN})^{2(3-i)+2(3-j)} \cdot Q^i U^j$ will induce hierarchical Yukawa couplings once X_{FN} develops a suitable vev. Similar considerations apply for the remaining Yukawa matrices.

On the other hand, the three generations all possess the same gauge quantum numbers with respect to the Standard Model gauge group. For this reason, introducing an horizontal symmetry may appear somewhat ad hoc. In this note we show that an approximate symmetry of the local geometry plays a similar role to that present in the Froggatt-Nielsen mechanism. Nevertheless, we also find that the exact structure of the Yukawa matrices differs from single field Froggatt-Nielsen models due to the profiles of the internal wave functions.

II. PARTIALLY TWISTED GAUGE THEORY AND F-THEORY GUTS

In F-theory GUT models (see [17, 18] for reviews), the gauge fields of the MSSM descend from the eight-dimensional worldvolume of a seven-brane wrapping a complex del Pezzo surface S with a GUT gauge group. This GUT group is broken to $SU(3)_C \times SU(2)_L \times U(1)_Y$ via an internal flux through the seven-brane in the $U(1)_Y$ direction of the GUT group [2, 9]. The chiral matter of the MSSM descends from zero modes of six-dimensional fields localized on matter curves in S . In the classical limit, the zero mode wave functions vanish off of the matter curve. Strictly speaking, this is only approximately true because these wave functions have non-vanishing support in directions transverse to the matter curve. In purely gauge theoretic terms, these wave functions can be derived from an appropriate internal local Higgsing of a parent eight-dimensional theory [20]. In this parent theory, there is an adjoint-valued field ϕ . When ϕ develops a vev which breaks the parent theory gauge group, adjoint-valued fermions in the parent eight-dimensional gauge multiplet descend to “bifundamentals” which are trapped along complex codimension one matter curves defined by the vanishing locus $\phi = 0$, much as in the Nielsen-Olesen vortex [19]. The trapped zero modes obey the system of equations [1]:

$$\omega_S \wedge \partial_A \psi_\alpha + \frac{i}{2} [\bar{\phi}, \chi_\alpha] = 0 + O\left(\frac{M_{GUT}}{M_*}\right) \quad (7)$$

$$\bar{\partial}_A \chi_\alpha - [\phi, \psi_\alpha] = 0 + O\left(\frac{M_{GUT}}{M_*}\right) \quad (8)$$

where ω_S denotes the Kähler form on S , and χ_α as well as ψ_α denote fermions in the 8d gauge multiplet. The subscript A reflects the background internal gauge field from the GUT seven-brane, as well as potentially other seven-branes which intersect the GUT seven-brane. In addition the “ $O\left(\frac{M_{GUT}}{M_*}\right)$ ” reflect the possibility of contributions from higher dimension operators induced by other background fluxes of the compactification associated with more general p-form potentials. Here, M_* is the characteristic mass scale of the F-theory compactification which is related to the GUT scale M_{GUT} and α_{GUT} as [2]:

$$M_*^4 = \alpha_{GUT}^{-1} \cdot M_{GUT}^4. \quad (9)$$

In general, the effects of these background fluxes can non-trivially mix with such higher-form potentials. For example, in an abelian gauge theory on a D-brane, the presence of an NS B-field shifts the field strength as:

$$F' = F + B. \quad (10)$$

For simplicity of presentation, consider the case where the background field strength is parallel to the matter curve. Letting z_\perp denote the coordinate normal to the curve $\phi = 0$, and z the coordinate on the curve $\phi = 0$, the trapped zero mode wave function is of the form:

$$\Psi \sim f(z, \bar{z}) \exp(-\gamma z_\perp \bar{z}_\perp), \quad (11)$$

where f is a $\bar{\partial}_A$ zero mode of the bundle defined by the gauge field along the curve $\phi = 0$, and γ depends on the overall scale specified by the Kähler form. Note that the ratio of any two zero modes of $\bar{\partial}_A$ can be expressed purely in terms of holomorphic functions of z , a fact that will be crucial in the analysis to follow. Further note that by rescaling the Kähler form so that $\gamma \rightarrow \infty$, the choice of representative used in the topological field theory can be made arbitrarily peaked along the matter curve.

The intersection of three matter curves in S leads to Yukawa couplings among the corresponding matter fields. The Yukawa couplings between the Higgs fields and chiral matter are given by triple overlap integrals of the form:

$$\lambda^{ij} = \int_S \Lambda \Psi^i \Phi^j. \quad (12)$$

Here, $\Lambda \Psi^i \Phi^j$ may be viewed as shorthand for the triple product of the wave functions $\Lambda_{H_u} \Psi_Q^i \Psi_U^j$, $\Lambda_{H_d} \Psi_Q^i \Psi_D^j$ or $\Lambda_{H_d} \Psi_L^i \Psi_E^j$ in the obvious notation. In the limit where the internal field variations (other than the Higgs) is constant, this integral reduces to a product of the three wave functions at the various intersection points of the matter curves:

$$\lambda^{ij} = \sum_p \Lambda(p) \Psi^i(p) \Phi^j(p) \quad (13)$$

where p denotes a point of triple intersection. Indeed, this superpotential is the descendant of the superpotential for bulk modes in the parent theory which makes no reference to the Kähler form. As such, we are free to rescale the Kähler form dependence until the overlap of the corresponding wave functions is only non-zero at the point of mutual intersection.

This is quite analogous to the computation of perturbative Yukawa couplings in the context of $(2,2)$ sigma models, where in the classical limit the Yukawa couplings are given by classical intersection theory. In the A-model, this result is then deformed by worldsheet instanton corrections and in the B-model by the variation of the complex structure. The setup in F-theory is parallel to the B-model setup and we can view the above Yukawa coupling as the analog of the limit of “large” complex structure. Just as in the B-model at finite complex structure, there will also be subleading corrections to equation (13). Such corrections can occur as a result of higher dimension operator contributions to the effective action, which will deform equations (7) and (8).

From a bottom up perspective, a minimal implementation of the chiral matter of an $SU(5)$ GUT requires one matter curve for the 10_M and one for the $\bar{5}_M$ where $SU(5)$ respectively enhances to $SO(10)$ and $SU(6)$. The Higgs 5_H and $\bar{5}_H$ localize on distinct curves where $SU(5)$ enhances to $SU(6)$. The background flux through these curves dictates the number of chiral matter zero modes in the low energy theory, while a suitable background flux through the Higgs curves achieves doublet-triplet splitting [2].

The superpotential terms $5_H \times 10_M \times 10_M$ and $\bar{5}_H \times \bar{5}_M \times 10_M$ respectively originate from enhancements to E_6 and $SO(12)$. Geometrically, this requires the 10_M curve to intersect itself, as well as the 5_H curve at a single point, and for the 10_M curve to intersect the $\bar{5}_M$ and $\bar{5}_H$ curves at another point of S [2]. We shall respectively label these triple intersection points as p_{up} and p_{down} . While additional intersections are in principle possible, the minimal, and most generic situation will typically only contain one intersection point of each type. Note that in principle there can be additional points where the singularity type enhances since the matter curve may consist of more than one irreducible component. In the special case where the wave functions are given identically by a sharply peaked Gaussian which is concentrated along a matter curve, a single triple intersection for each type of Yukawa leads to rank one Yukawa matrices [2]. For example, the up quark Yukawas are given by:

$$\lambda_u^{ij} = \Lambda_{H_u}(p_{up}) \Psi_Q^i(p_{up}) \Psi_U^j(p_{up}) \quad (14)$$

which is manifestly a rank one matrix. If there were no further corrections, the resulting theory would contain a massive top quark, and massless up and charm quarks. Similar considerations hold for the down type and lepton masses.

In evaluating subleading corrections to the Yukawas, it is more natural to go to a canonical orthonormal basis of wave functions where this rank one structure in the Yukawa matrix is more manifest. To this end, let z_1 and z_2 denote two local coordinates on S such that $z_1 = 0$ and $z_2 = 0$ denote the matter curves where the wave functions Φ^j and Ψ^i of equation (12) respectively localize. By an appropriate unitary change of basis, we can always arrange for $\Psi^i \sim z_1^{3-i}$ and $\Phi^j \sim z_2^{3-j}$ along the respective curves where the wave functions localize (in particular we can choose these wave functions to be orthonormal by allowing subleading pieces in the powers of z_i in each wave function). In this basis, $\lambda^{33} \neq 0$ whereas all the other components of λ^{ij} vanish. In terms of the wave function overlap integral of equation (12) this vanishing structure in the Yukawa matrix can be ascribed to the presence of a geometric $U(1)_1 \times U(1)_2$ action on the z_j :

$$z_j \rightarrow \exp(i\alpha_j) z_j. \quad (15)$$

The terms of the integrand which vanish are precisely those which are not invariant under this $U(1)_1 \times U(1)_2$ symmetry [21].

III. WAVE FUNCTION DISTORTION AND YUKAWA HIERARCHIES

We now compute subleading corrections in the Yukawa matrices due to distortions in the profile of the internal wave functions. In principle, such distortions from various fluxes induced by p-form potential of the compactification. The ones of relevance for distorting the Yukawas correspond to higher dimension operator deformations of equations (7) and (8). The overlap integral of interest is given by:

$$\lambda^{ij} = \int d^2 z_1 d^2 z_2 \cdot \Omega(z_1, \bar{z}_1, z_2, \bar{z}_2) \cdot \left(\frac{z_1}{R_1}\right)^{3-i} \left(\frac{z_2}{R_2}\right)^{3-j} \quad (16)$$

where R_1 and R_2 denote the characteristic lengths of the curves $z_1 = 0$ and $z_2 = 0$, and $\Omega(z_1, \bar{z}_1, z_2, \bar{z}_2)$ is a function which includes the contributions from the Higgs field wave function, as well as other non-holomorphic terms associated with the other wave functions. Note that all information about flavor is contained in the term $z_1^{3-i} z_2^{3-j}$.

As noted above, a non-zero contribution to λ^{ij} can only occur when a contribution to the integrand is invariant under the local $U(1)_1 \times U(1)_2$ action of the coordinates. A non-zero entry for the λ^{ij} when i and j are both different from 3 therefore requires some contribution from Ω to carry non-trivial $U(1)_1 \times U(1)_2$ charge.

Such corrections originate from general fluxes of the compactification. Since ϕ is holomorphic and in the local geometry lies in a $U(1) \times U(1)$ subgroup of the parent local gauge symmetry, without loss of generality we can choose local coordinates so that $\phi_1 = z_1$ and $\phi_2 = z_2$ in the neighborhood of the intersection point. Our aim will be to characterize possible violations of the $U(1)_1 \times U(1)_2$ symmetry due to background fluxes. Owing to the symmetries of equations (7) and (8) and their deformation by possible higher dimension operators, the leading order distortion of the wave functions from background fluxes is a linear combination of the $U(1)_1 \times U(1)_2$ invariants $z_i \bar{z}_j$ so that:

$$\Psi = \Psi^{(0)} \cdot \exp\left(\mathcal{M}^{i\bar{j}} z_i \bar{z}_j\right) \quad (17)$$

where $\Psi^{(0)}$ denotes the contribution to the wave function in the limit where the field strength vanishes, and $\mathcal{M}^{i\bar{j}}$ is a 2×2 matrix. Although the off-diagonal contributions to this quadratic form indeed violate the ‘‘axial’’ combination of $U(1)_1$ and $U(1)_2$, such terms cannot contribute to the Yukawa couplings [22].

We stress that the distortion due to \mathcal{M} can in principle originate not just from gauge fields, but from more general fluxes induced by p-form potentials of the compactification, which will also mix with these gauge fields. Indeed, while gauge field flux alone does not turn out to distort the Yukawa couplings, other fluxes do [26]. Further note that by dimensional analysis, the overall scaling of \mathcal{M} is fixed as $\mathcal{M} \sim M_{GUT}^2$.

More generally, the field strengths will vary over points of the geometry. In an adiabatic approximation we can simply retain the form of Ψ but with $\mathcal{M}^{i\bar{j}}$ now given by a function of z_1, \bar{z}_1, z_2 and \bar{z}_2 . We now perform a series expansion in derivatives of $\mathcal{M}^{i\bar{j}}$ around the point of triple intersection:

$$\mathcal{M}^{i\bar{j}} z_i \bar{z}_j = \sum_{k,l,m,n} \frac{\partial_1^k \bar{\partial}_1^l \partial_2^m \bar{\partial}_2^n \mathcal{M}^{i\bar{j}}(0)}{k!l!m!n!} \cdot z_1^k \bar{z}_1^l z_2^m \bar{z}_2^n \cdot z_i \bar{z}_j. \quad (18)$$

The crucial point is that the monomial $z_1^k \bar{z}_1^l z_2^m \bar{z}_2^n$ has $U(1)_1 \times U(1)_2$ charge $(k-l, m-n)$. Expanding the exponential of equation (17), it now follows that there will generically be corrections to the rank one Yukawa coupling.

We now compute the leading order behavior of the Yukawa matrix entry λ^{ij} by performing a series expansion of Ω in equation (16). $U(1)_1 \times U(1)_2$ charge conservation implies that the only terms from a series expansion of Ω which can contribute are of the form:

$$(z_1 \bar{z}_1)^a (z_2 \bar{z}_2)^b \cdot (\bar{z}_1)^{3-i} (\bar{z}_2)^{3-j}. \quad (19)$$

Each power of z reflects the presence of a derivative in the series expansion.

There are in principal two expansion parameters available which will lead to different hierarchies in the Yukawa couplings. The first, and perhaps most obvious possibility is that we can simply count the number of total z ’s which appear in a given term. This amounts to performing a derivative expansion in the gauge field strength. In this case, the leading order contributions will always originate from terms where $a = b = 0$. The contribution from the derivative expansion is given as:

$$\delta \lambda_{DER}^{ij} = c^{ij} \cdot \frac{\bar{\partial}_1^{3-i} \bar{\partial}_2^{3-j} \mathcal{M}(0)}{M_*^{8-i-j}} \cdot \left(\frac{1}{R_1 M_*}\right)^{3-i} \left(\frac{1}{R_2 M_*}\right)^{3-j} \quad (20)$$

where \mathcal{M} is shorthand for the generic entry \mathcal{M}^{ij} , c^{ij} is an order one coefficient which reflects the evaluation of the integral in equation (16).

On the other hand, there is another natural expansion parameter given by expanding in successive powers of the first gradient of the flux. The leading order perturbation to the Yukawas from this class of terms is:

$$\delta\lambda_{FLX}^{ij} = c^{ij} \cdot \left(\frac{\bar{\partial}_1 \mathcal{M}(0)}{R_1 M_*^4} \right)^{3-i} \left(\frac{\bar{\partial}_2 \mathcal{M}(0)}{R_2 M_*^4} \right)^{3-j}. \quad (21)$$

To estimate the magnitude of each $\delta\lambda^{ij}$, we note by dimensional analysis, $\mathcal{M} \sim M_{GUT}^2$. Further, each successive derivative of \mathcal{M} contributes an extra factor of $1/R_1$ or $1/R_2$. Setting:

$$\varepsilon_0 = \mathcal{M}(0)/M_*^2, \varepsilon_1 = (M_* R_1)^{-2}, \varepsilon_2 = (M_* R_2)^{-2}, \quad (22)$$

we thus obtain:

$$\delta\lambda_{DER}^{ij} = c^{ij} \cdot \varepsilon_0 \cdot (\varepsilon_1)^{3-i} (\varepsilon_2)^{3-j} \quad (23)$$

$$\delta\lambda_{FLX}^{ij} = c^{ij} \cdot (\varepsilon_0 \varepsilon_1)^{3-i} (\varepsilon_0 \varepsilon_2)^{3-j} \quad (24)$$

adopting a parametrization where $\varepsilon_1 \sim \varepsilon_2 \sim \kappa$ and $\varepsilon^2 \sim \varepsilon_0 \varepsilon_2$, the Yukawa coupling matrix therefore assumes the form:

$$\lambda = \lambda_{(0)} + \delta\lambda_{DER} + \delta\lambda_{FLX} \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \kappa^3 \cdot \varepsilon^2 & \kappa^2 \cdot \varepsilon^2 & \kappa \cdot \varepsilon^2 \\ \kappa^2 \cdot \varepsilon^2 & \kappa \cdot \varepsilon^2 & \varepsilon^2 \\ \kappa \cdot \varepsilon^2 & \varepsilon^2 & \varepsilon_0 \end{pmatrix} + \begin{pmatrix} \varepsilon^8 & \varepsilon^6 & \varepsilon^4 \\ \varepsilon^6 & \varepsilon^4 & \varepsilon^2 \\ \varepsilon^4 & \varepsilon^2 & 1 \end{pmatrix} \quad (25)$$

where each entry is multiplied by an order one constant. At this point, it is important to note that although the analysis crucially depends on the violation of a $U(1)$ symmetry, the structure of $\delta\lambda_{DER}$ is somewhat different from the simplest Froggatt-Nielsen parametrization mentioned earlier. Note, however, that $\delta\lambda_{FLX}$ is indeed consistent with a single field Froggatt-Nielsen model.

In the geometry, there are in fact three κ 's and three ε 's because there are three distinct Yukawa matrices of interest. Indeed, the background hyperflux will generate order one distortions in the lepton doublet and down quark wave functions so that the entries of λ_l and λ_d can differ by order one constants, so that there is no a priori mass relation between the down type quarks and the charged leptons [2].

Because $R_i \sim M_{GUT}^{-1}$, one might at first think that since $\varepsilon \sim \kappa \sim \sqrt{\alpha_{GUT}}$, that the DER expansion will always dominate over the FLX expansion. However, there can be a further enhancement in the FLX expansion because it is more strongly on how chiral matter couples to background fluxes. Since $\sqrt{\alpha_{GUT}}$ is not extremely small, the FLX expansion can in principle dominate over the DER expansion in Yukawas involving fields which have large couplings to the background flux.

Determining whether $\delta\lambda_{DER}$ or $\delta\lambda_{FLX}$ dominates at each order depends on how the left and right chiral matter wave functions couple to the background fluxes. In F-theory GUTs, a background flux in the hypercharge direction is always present [2, 9], so to illustrate this point, consider the coupling of the matter wave functions to this flux. We shall assume that the strength of this coupling to background fluxes holds for other more general fluxes since in general, these higher form fluxes can mix with the gauge fields, as in equation (10) so we shall use it as a rough guide to determine which type of expansion dominates for a given type of particle species. In the case of hyperflux, the internal wave functions couple to this particular background flux in strength proportional to their hypercharges. It follows that the hyperflux contribution to \mathcal{M}^{ij} is proportional to the corresponding hypercharge. The term $\delta\lambda_{FLX}^{ij}$ can dominate over $\delta\lambda_{DER}^{ij}$ provided:

$$\left(Y_{\max} \cdot \frac{\bar{\partial} F(0)}{R M_*^4} \right)^2 > Y_{\max} \cdot \frac{\bar{\partial}^2 F(0)}{M_*^4} \cdot \left(\frac{1}{R M_*} \right)^2 \quad (26)$$

where F denotes the generic field strength, and Y_{\max} denotes in an integral normalization the maximal norms of the hypercharge for the left and right chiral matter wave functions associated to each Yukawa. In particular, we have:

	lepton	up	down
Y_{\max}	6	4	2

(27)

Using the representative scaling $F/M_*^2 \sim 1/(R M_*)^2 \sim \alpha_{GUT}^{1/2}$ yields:

$$\mathcal{F} \equiv Y_{\max} \cdot \alpha_{GUT}^{1/2} \gtrsim 1, \quad (28)$$

where \mathcal{F} measures the relative strengths of the flux and gradient expansions. Plugging in $\alpha_{GUT}^{1/2} \sim 0.2$ as well as the integral values for the hypercharges yields:

	lepton	up	down
\mathcal{F}	$1.2 \sim 1$	$0.8 \sim 1$	$0.4 < 1$

(29)

This rough computation suggests that for the down type Yukawas, $\delta\lambda_{DER}$ dominates, whereas for the up type and charged lepton Yukawas, $\delta\lambda_{FLX}^j$ could potentially dominate. Even though this is by itself quite heuristic, we will shortly find that comparison with observation indeed corroborates this picture.

Setting $\kappa \sim \varepsilon$ the Yukawa matrices for λ_{DER} and λ_{FLX} are:

$$\lambda_{DER} \sim \begin{pmatrix} \varepsilon^5 & \varepsilon^4 & \varepsilon^3 \\ \varepsilon^4 & \varepsilon^3 & \varepsilon^2 \\ \varepsilon^3 & \varepsilon^2 & 1 \end{pmatrix}, \quad \lambda_{FLX} \sim \begin{pmatrix} \varepsilon^8 & \varepsilon^6 & \varepsilon^4 \\ \varepsilon^6 & \varepsilon^4 & \varepsilon^2 \\ \varepsilon^4 & \varepsilon^2 & 1 \end{pmatrix} \quad (30)$$

this leads to distinct mass hierarchies:

$$\lambda_{DER} \Rightarrow m_1 : m_2 : m_3 \sim \varepsilon^5 : \varepsilon^3 : 1 \quad (31)$$

$$\lambda_{FLX} \Rightarrow m_1 : m_2 : m_3 \sim \varepsilon^8 : \varepsilon^4 : 1. \quad (32)$$

Next consider diagonalization of λ via the matrices V^L and V^R . In fact, it is enough to diagonalize $\lambda\lambda^\dagger$:

$$(\lambda_{DER})(\lambda_{DER})^\dagger \sim \begin{pmatrix} \varepsilon^6 & \varepsilon^5 & \varepsilon^3 \\ \varepsilon^5 & \varepsilon^4 & \varepsilon^2 \\ \varepsilon^3 & \varepsilon^2 & 1 \end{pmatrix}, \quad (\lambda_{FLX})(\lambda_{FLX})^\dagger \sim \begin{pmatrix} \varepsilon^8 & \varepsilon^6 & \varepsilon^4 \\ \varepsilon^6 & \varepsilon^4 & \varepsilon^2 \\ \varepsilon^4 & \varepsilon^2 & 1 \end{pmatrix}. \quad (33)$$

Note that these matrices have a structure of the form $\varepsilon^{a_i+a_j}$ which is consistent with a single field Froggatt-Nielsen model with charges 3, 2, 0 for $(\lambda_{DER})(\lambda_{DER})^\dagger$ and 4, 2, 0 for $(\lambda_{FLX})(\lambda_{FLX})^\dagger$. In this case, the matrices V^L and V^R are of the form $\varepsilon^{|a_i-a_j|}$ so that:

$$V_{DER}^L \sim V_{DER}^R \sim \begin{pmatrix} 1 & \varepsilon & \varepsilon^3 \\ \varepsilon & 1 & \varepsilon^2 \\ \varepsilon^3 & \varepsilon^2 & 1 \end{pmatrix}, \quad V_{FLX}^L \sim V_{FLX}^R \sim \begin{pmatrix} 1 & \varepsilon^2 & \varepsilon^4 \\ \varepsilon^2 & 1 & \varepsilon^2 \\ \varepsilon^4 & \varepsilon^2 & 1 \end{pmatrix}. \quad (34)$$

In the above we have focussed on the overlap of the wave functions at a single point of the geometry. At each intersection point, there is a local choice of $U(1)$'s with respect to which the given mass hierarchy of the generations is manifest. Indeed, this is a basis independent statement.

On the other hand, the CKM matrix $V_{CKM} = V_u^L V_d^{L\dagger}$ measures the mismatch between the matrices which diagonalize the Yukawas in the *same* basis. Preserving the hierarchy of the CKM matrix requires that the hierarchy present in V_u^L and V_d^L is manifest in the same basis for the Q wave functions. For this to be the case, the eigenspace decomposition defined by the local $U(1)$'s at the two different interaction points must remain relatively unchanged. This occurs provided the distance between the intersection points p_{up} and p_{down} (both of which lie on the Q matter curve) should be less than $0.1 \times M_{GUT}^{-1}$. Note that this is to be compared with the length scale of the curves, which is roughly M_{GUT}^{-1} . While this is certainly possible, and involves only a very mild fine tuning in the location of the two points of triple intersection, it is suggestive of a higher unification structure which could naturally accommodate the strict identification $p_{up} = p_{down}$. The presence of a higher unification structure is also in line with the appearance of a $U(1)_{PQ}$ factor in F-theory GUT scenarios which incorporate the effects of supersymmetry breaking [3]. For the purposes of this paper, we shall only assume that p_{up} and p_{down} are sufficiently close that this subtlety can be neglected.

Under this mild assumption, it is in fact now possible to determine the form of the CKM matrix. When either V^L matrix is determined by the derivative expansion, $\kappa \sim \varepsilon$ dominates over order ε^2 terms so that:

$$V_{CKM}^{F-th}(\varepsilon) \sim \begin{pmatrix} 1 & \varepsilon & \varepsilon^3 \\ \varepsilon & 1 & \varepsilon^2 \\ \varepsilon^3 & \varepsilon^2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & \alpha_{GUT}^{1/2} & \alpha_{GUT}^{3/2} \\ \alpha_{GUT}^{1/2} & 1 & \alpha_{GUT} \\ \alpha_{GUT}^{3/2} & \alpha_{GUT} & 1 \end{pmatrix}. \quad (35)$$

By inspection, the 3, 2, 0 Froggatt-Nielsen charges dominate in the CKM matrix. In addition, the gap in the Froggatt-Nielsen charges can be traced back to the tensor structure of the flux, which distinguishes it from a scalar. In the less likely possibility (which is indeed not realized) where both V^L 's are given by V_{FLX}^L , the form of V_{CKM} is again of the form V_{FLX}^L .

IV. COMPARISON WITH EXPERIMENT

In the previous section we obtained a parametrization of the up, down and lepton Yukawas in terms of the parameters ε and κ . In this section, we show that the most natural estimates for ε and κ dictated by the GUT structure are in beautiful accord with observation. Using the explicit values:

$$\kappa \sim \varepsilon \sim M_{GUT}^2/M_*^2 \sim \alpha_{GUT}^{1/2} \sim 0.2, \quad (36)$$

we can now estimate the CKM matrix to be:

$$V_{CKM}^{F-th} \sim \begin{pmatrix} 1 & \alpha_{GUT}^{1/2} & \alpha_{GUT}^{3/2} \\ \alpha_{GUT}^{1/2} & 1 & \alpha_{GUT} \\ \alpha_{GUT}^{3/2} & \alpha_{GUT} & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0.2 & 0.008 \\ 0.2 & 1 & 0.04 \\ 0.008 & 0.04 & 1 \end{pmatrix}. \quad (37)$$

Comparing with the measured values [11]:

$$|V_{CKM}(M_{weak})| \sim \begin{pmatrix} 0.97 & 0.23 & 0.004 \\ 0.23 & 0.97 & 0.04 \\ 0.008 & 0.04 & 0.99 \end{pmatrix}, \quad (38)$$

reveals a beautiful match between theory and observation! Even though we should only have expected an agreement in parameters up to order one effects, we find this close match with the data reassuring.

We also find close agreement with the expected mass hierarchies in the quark and charged lepton sectors. Due to the different structure of the perturbations $\delta\lambda_{DER}$ and $\delta\lambda_{FLX}$, there are in principle two different hierarchical mass ratios given by:

$$\lambda_{DER} \Rightarrow m_1 : m_2 : m_3 \sim \varepsilon^5 : \varepsilon^3 : 1 \quad (39)$$

$$\lambda_{FLX} \Rightarrow m_1 : m_2 : m_3 \sim \varepsilon^8 : \varepsilon^4 : 1. \quad (40)$$

As discussed previously, the dominant contribution to λ_d is expected to be from $\delta\lambda_{FLX}$. On the other hand, we have observed that the dominant contribution to λ_u and λ_l could in principle be either $\delta\lambda_{DER}$ or $\delta\lambda_{FLX}$.

To compare with observation, we can fit the observed masses to a given value of ε . At the present crude level of analysis, it is enough to neglect the effects of running in such a match because we will be taking third and fourth roots of order one numbers. It turns out that the best match to our scenario is achieved when both λ_u and λ_l are of the latter type with corresponding values for ε :

	l_{FLX}	u_{FLX}	d_{DER}
ε	0.36	0.26	0.27

(41)

which are all within order one factors of $\alpha_{GUT}^{1/2} \sim 0.2$! Fixing the values of the top, bottom and tau mass to their observed values, we can now use the ratios of (39) and (40) to extract the masses of the lighter generations. Comparing the F-theory result with the observed values yields:

$$(m_u, m_u^{F-th}), (m_c, m_c^{F-th}), (m_t, m_t^{F-th}) \sim (0.003, 0.004), (1.3, 0.8), (170, 170) \times \text{GeV} \quad (42)$$

$$(m_d, m_d^{F-th}), (m_s, m_s^{F-th}), (m_b, m_b^{F-th}) \sim (0.005, 0.006), (0.1, 0.08), (4, 4) \times \text{GeV} \quad (43)$$

$$(m_e, m_e^{F-th}), (m_\mu, m_\mu^{F-th}), (m_\tau, m_\tau^{F-th}) \sim (0.0005, 0.0005), (0.1, 0.03), (1.8, 1.8) \times \text{GeV}, \quad (44)$$

which by inspection, are quite similar.

The parametrization of the up and charged lepton Yukawas in terms of $\delta\lambda_{DER}$ does not reliably reproduce these same mass ratios. One way to see this is to restore the explicit ε and κ dependence in the mass ratios. There are two independent ratios of masses, so in this case we can exactly solve for ε and κ with the result:

	l_{DER}	u_{DER}	d_{DER}
ε	0.88	0.40	0.33
κ	0.07	0.05	0.22

(45)

Note that the ε 's for l_{DER} and u_{DER} are both greater than $\alpha_{GUT}^{1/2}$, whereas the κ 's are both smaller. Indeed, the ratio ε/κ for these two cases is now an order ten number, rather than the order one parameter expected from general

considerations. Note, however, that the ratio ε/κ for d_{DER} is indeed an order one number, which is in accord with our scenario.

The match to the CKM matrix and the mass parametrization of table (41) lends considerable credence to the simple picture of wave function distortion we have found. We note that although we have implicitly worked within the framework of the MSSM, the hierarchies we have found are more general, and only require a supersymmetric structure near the GUT scale.

Up to now, we have only discussed the hierarchy in the mass ratios. In fact, our model also predicts the actual masses for the top, bottom and tau mass as well [2]:

$$m_t^{F-th}(M_{GUT}) \sim \alpha_{GUT}^{3/4} \cdot \langle H_u \rangle = \alpha_{GUT}^{3/4} \cdot \langle H \rangle \sin \beta \quad (46)$$

$$m_b^{F-th}(M_{GUT}) \sim m_\tau^{F-th}(M_{GUT}) = \alpha_{GUT}^{3/4} \cdot \langle H_d \rangle = \alpha_{GUT}^{3/4} \cdot \langle H \rangle \cos \beta. \quad (47)$$

Using the value $\langle H \rangle \sim 170$ GeV, and assuming the large value $\tan \beta \sim 30$ common to the models discussed in [3] yields:

$$m_t^{F-th}(M_{GUT}) \sim 20 \text{ GeV} \quad (48)$$

$$m_b^{F-th}(M_{GUT}) \sim m_\tau^{F-th}(M_{GUT}) \sim 0.6 \text{ GeV}, \quad (49)$$

which are to be compared with the observed values run up to the GUT scale. These running effects are roughly given by a factor of three reduction for the quark masses so that:

$$m_t(M_{GUT}) \sim 55 \text{ GeV} \quad (50)$$

$$m_b(M_{GUT}) \sim m_\tau(M_{GUT}) \sim 1.5 \text{ GeV} \quad (51)$$

which matches the values of our scenario up to order one factors (in fact, a factor of three). In tandem with our estimate for the mass ratios, we thus obtain a crude estimate for the masses of all the quarks and charged leptons in terms of $\langle H \rangle$, $\tan \beta$ and α_{GUT} .

V. DISCUSSION

In this note we have shown that simply demanding a local F-theory GUT model with the most generic features automatically implies hierarchical structure in the Yukawa couplings. Moreover, we have found that simple estimates on the behavior of the wave functions near points of triple intersection are in remarkable accord with observation! While it is certainly quite satisfying to see this type of structure naturally emerge from a string based model, it is also possible to go further.

The first point is that the general form of the matrices we have found can in principle be extended to an arbitrary number of generations. Indeed, it is interesting to ask whether this model constrains the existence of additional generations of quarks [23]. For example, in a four generation model, the analogue of equation (35) is now determined by the FN charges 4, 3, 2, 0, leading to:

$$V_{CKM}^{(4-gen)} \sim \begin{pmatrix} 1 & \varepsilon & \varepsilon^2 & \varepsilon^4 \\ \varepsilon & 1 & \varepsilon & \varepsilon^3 \\ \varepsilon^2 & \varepsilon & 1 & \varepsilon^2 \\ \varepsilon^4 & \varepsilon^3 & \varepsilon^2 & 1 \end{pmatrix}. \quad (52)$$

Mixing in the three lightest generations corresponds to the upper left 3×3 block of this matrix, which is in worse agreement with experiment than the three generation model. Thus, we conclude that independent of other experimental constraints, simply obtaining the observed CKM matrix structure for the three lightest generations in our model *predicts* that there is no fourth generation of quarks!

We have also seen that the generic profile of the internal wave functions generates a rank three Yukawa matrix. In particular, this implies that the up quark is not massless, in accord with results from lattice gauge theory simulations of QCD. In particular, this implies that the strong CP problem is real. Interestingly, supersymmetry breaking in F-theory GUT models automatically contain an axion with a phenomenologically viable value for the decay constant [2].

We find it quite remarkable that the most minimal geometric structures in this framework are in fact sufficient for the purposes of achieving phenomenologically viable hierarchies in the Yukawa couplings. Turning the analysis

around, the beautiful match obtained in minimal F-theory GUT scenarios can be viewed as an important constraint on the class of compactifications, and in particular the requisite geometries which can reproduce detailed features of the MSSM.

Finally, one can extend this analysis to the case of neutrino masses and mixing. Whereas the quarks and charged leptons localize on matter curves in S , in some F-theory GUT models the right-handed neutrinos N_R localize on matter curves which only touch S at a point [2]. We have argued that the Yukawa matrices for the quarks and charged leptons can exhibit similar hierarchies provided the two points of intersection p_{up} and p_{down} are sufficiently close together. Assuming appropriate wave function overlaps so that the superpotential terms $LN_R H_u$ and $M_{maj} N_R N_R$ are present will generate viable masses for the neutrinos via the seesaw mechanism. Note, moreover, that this type of geometry can also easily accommodate large mixing angles provided the interaction point leading to LEH_d on the L -matter curve is not finely tuned to be near the interaction point leading to $LN_R H_u$. We are currently working out the details of this scenario.

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VI. APPENDIX: WAVE FUNCTION DISTORTION FROM FLUXES

In this Appendix we provide additional details on the profile of wave functions in the presence of a background gauge field strength. We refer the interested reader to [2] for a more detailed analysis of wave function distortion effects due to gauge field strengths in the direction parallel to the matter curve. We present this example to illustrate how wave function distortion works in general. Indeed, it turns out that though gauge field fluxes alone do not distort the profile of the Yukawas, these contributions can mix with higher form potentials, leading to an appropriate Yukawa distortion [26].

We first consider the case where this gauge field strength is constant, and then explain how the profile of the wave function changes in the presence of variations in this field. To be completely explicit, we consider the case of a bulk E_6 theory with 27's localized along three matter curves of rank one enhancement to E_7 which form a triple intersection at a point where E_6 enhances to E_8 . In the parent E_8 theory, the background values of the scalar ϕ and internal gauge fields satisfy the BPS equations of motion [1]:

$$\bar{\partial}_A \phi = 0, F^{(2,0)} = 0, F^{(0,2)} = 0, \omega_S \wedge F^{(1,1)} = 0. \quad (53)$$

The E_6 theory is defined by Higgsing the parent E_8 theory in the direction: $\langle \phi \rangle = \langle \phi_1 \rangle T_1 + \langle \phi_2 \rangle T_2$ where the T_i denote two Cartan generators in the $SU(3)$ factor of $E_6 \times SU(3) \subset E_8$. The loci of partial enhancement to E_7 are then given by $\langle \phi_1 \rangle = 0$, $\langle \phi_2 \rangle = 0$ and $\langle \phi_1 \rangle + \langle \phi_2 \rangle = 0$. In addition, we also consider a background gauge field strength taking values in the Cartan of $SU(3)$.

Adopting a local system of coordinates z_1, z_2 such that $z_i = \langle \phi_i \rangle$, the modes trapped along the curve $z_1 = 0$ curve satisfy equations (7) and (8):

$$v_2 (\partial_1 + A_1) \psi_{\bar{1}} + v_1 (\partial_2 + A_2) \psi_{\bar{2}} + \bar{z}_1 \chi_{12} = 0 \quad (54)$$

$$(\bar{\partial}_{\bar{1}} + A_{\bar{1}}) \chi_{12} + z_1 \psi_{\bar{1}} = 0 \quad (55)$$

$$(\bar{\partial}_{\bar{2}} + A_{\bar{2}}) \chi_{12} + z_1 \psi_{\bar{2}} = 0 \quad (56)$$

where in the above, we have taken a canonical presentation for the Kähler form. Similar equations hold for the other massless modes of the theory. When the background field strength is constant, the solutions to this system of equations depend to leading order on the fluxes through a term of the form $\exp(\mathcal{M}^{i\bar{j}} z_i \bar{z}_{\bar{j}})$, where $\mathcal{M}^{i\bar{j}}$ depends on the background field strength configuration.

As an example, consider the special case where only $F_{1\bar{1}}$ and $F_{2\bar{2}}$ are non-zero. A gauge field configuration which reproduces this field strength is:

$$A = -F_{1\bar{1}} \bar{z}_1 \cdot dz_1 + F_{2\bar{2}} z_2 \cdot d\bar{z}_2. \quad (57)$$

Solving equations (54)-(56) yields $\psi_{\bar{2}} = 0$ and:

$$\psi_{\bar{1}} \propto \chi_{12} \propto \alpha(z_2) \cdot \exp(-z_1 \bar{z}_1 / \sqrt{v_2}) \cdot \exp(-F_{2\bar{2}} \cdot z_2 \bar{z}_2 + (F_{1\bar{1}} \cdot z_1 \bar{z}_1 / 2) + O(\sqrt{v_2})) \quad (58)$$

where $\alpha(z_2)$ denotes a holomorphic function of z_2 . Adiabatically including the position dependence in the $F_{i\bar{j}}$ generates a series expansion in the z 's.

Note added in revised version:

In the context of minimal F-theory GUTs, the $5 \times 10 \times 10$ interaction term originates from a local enhancement from $SU(5)$ to E_6 . Here, the 5 localizes on a curve where $SU(5)$ enhances to $SU(6)$, and the 10's localize on curves where $SU(5)$ enhances to $SO(10)$. A subtle point in this construction is that when a full GUT multiplet localizes on the $SO(10)$ curve, achieving a rank one matrix requires the 10's to localize on the same matter curve [2]. This appears to suggest the condition that the curve on which the 10's localize should self-intersect, or “pinch” at the point of E_6 enhancement. However, as pointed out in [25], the wave functions of the Q fields will now have two components, Ψ_{Q+} and Ψ_{Q-} , corresponding to the profile of the Q wave functions on the two local pieces of the pinched curve. Combined with the analogous contribution from the U wave functions, this would then lead to a sum of two Yukawa couplings. Because Ψ_{Q+} and Ψ_{Q-} a priori have different coordinate dependence (as they localize on different pieces of the same pinched curve), this will generically lead to a rank two matrix, unlike the rank one case required in the analysis of this paper. A simple remedy to this issue is to impose a \mathbb{Z}_2 symmetry which interchanges these two components of the pinched curve. Alternatively, we can consider geometries quotiented by this same \mathbb{Z}_2 symmetry. In the quotiented geometry, the two enhancements from $SU(5)$ to $SO(10)$ are identified, with the E_6 point invariant under this group action. More generally, one can consider geometries where a \mathbb{Z}_2 action interchanges two smooth curves.

Although the existence of such \mathbb{Z}_2 quotients may appear ad hoc, this is in fact the *generic* situation in compactifications of F-theory! As noted in [1], there could be monodromies acting on the seven-branes of F-theory. In fact such monodromies were crucial for understanding how the non-simply laced groups arise from F-theory compactifications [24]. However, such brane monodromies were not used in the constructions of F-theory GUTs presented in [1]. As pointed out in [25], exactly these generic monodromies will make the \mathbb{Z}_2 quotient just described above automatic!

The essential point is that in the breaking pattern $E_6 \supset SU(5) \times SU(2) \times U(1)$, it is a rather special condition on the nature of the singularity to require two distinct $SO(10)$ matter curves. Indeed, typically there will be a branch cut in the configuration, so that the two seemingly distinct components where a 10 localizes will interchange under monodromy around the point of enhancement. This corresponds to a single smooth curve with a branch cut locus emanating out from the point of enhancement.

The relevance of such geometries for phenomenology was clarified in [25] where these configurations were analyzed as deformations of an element of the Cartan of E_6 such that the \mathbb{Z}_2 Weyl group of the $SU(2)$ factor in the above breaking pattern permutes the two 10's. This follows from the fact that the 10's transform as a doublet of $SU(2)$. This is the same geometric \mathbb{Z}_2 quotient discussed earlier, and shows that rather than being special, it is in fact the more generic case from the perspective of geometry.

It is important to note, however, that just as in any orbifold theory, working in terms of quantities invariant under the group action in the covering theory clearly suffices in all computations. Thus, it is always enough to compute all relevant wave function overlaps in the covering theory. Quotienting by the appropriate Weyl group (in the E_6 breaking to $SU(5)$ case, the Weyl group of $SU(2)$), it follows that the computation of Yukawas in the covering theory fully determines the Yukawas in the quotient theory, and does not modify the estimates presented in this paper.

Finally, it has also recently been found in [26] that background gauge field fluxes alone do not distort the rank of the Yukawa matrix. Nevertheless, the general philosophy of this paper that background fluxes of the compactification can alter the profile of the matter field wave functions, and also the structure of the Yukawa matrices was recently confirmed in [26] by considering a non-commutative deformation of the F-terms of the seven-brane superpotential induced by background $H_{(1,2)} = H_R + \tau_{IIB} H_{NS}$ fluxes of the compactification. Such contributions correspond to higher dimension operators of the type briefly alluded to in equations (7) and (8). In some simplified examples it was found in [26] that much of the hierarchical structure of the quark and charged lepton masses, and the CKM matrix are recovered.

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